Applications of differential subordination for functions with fixed second coefficient to geometric function theory

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ABSTRACT. The theory of first-order differential subordination developed by Miller and Mocanu was recently extended to functions with fixed initial coefficient by R. M. Ali, S. Nagpal and V. Ravichandran [Second-order differential subordination for analytic functions with fixed initial coefficient, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), 611–629] and applied to obtain several generalization of classical results in geometric function theory. In this paper, further applications of this subordination theory is given. In particular, several sufficient conditions related to starlikeness, convexity, close-to-convexity of normalized analytic functions are derived. Connections with previously known results are pointed out.

1. Introduction

For univalent functions $f(z) = z + a_2z^2 + \cdots$ defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the famous Bieberbach theorem shows that $|a_2| \leq 2$ and this bound for the second coefficient yields the growth and distortion bounds as well as covering theorem. In view of the influence of the second coefficient in the properties of univalent functions, several authors have investigated functions with fixed second coefficient. For a brief survey of the various developments, mainly on radius problems, from 1920 to this date, see the recent work by Ali *et al.* [**2**]. The theory of first-order differential subordination was developed by Miller and Mocanu and a very comprehensive account of the theory and numerous application can be found in their monograph [**6**]. Ali *et al.* [**3**] have extended this well-known theory of differential subordination to the functions with preassigned second coefficients. Nagpal and Ravichandran [**7**] have applied the results in [**3**] to obtain several extensions of well-known results to the functions with fixed second coefficient. In this paper, we continue their investigation by deriving several sufficient conditions for starlikeness of functions with fixed second coefficient.

For convenience, let $\mathscr{A}_{n,b}$ denote the class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$ where $n \in \mathbb{N}$ and b is a fixed non-negative real number. For a fixed $\mu \geq 0$, let $\mathscr{H}_{\mu,n}$ consists of analytic functions p on $\mathbb{D}$ of the form

$$p(z) = 1 + \mu z^n + p_{n+1}z^{n+1} + \cdots, \quad n \in \mathbb{N} \tag{1.1}$$

Let Ω be a subset of $\mathbb{C}$ and the class $\Psi_{\mu,n}[\Omega]$ consists of those functions $\psi : \mathbb{C}^2 \to \mathbb{C}$ that are continuous in a domain $D \subset \mathbb{C}^2$ with $(1,0) \in D$, $\psi(1,0) \in \Omega$, and satisfy the admissibility

2000 *Mathematics Subject Classification.* 30C45, 30C80.

Key words and phrases. Analytic functions, starlike functions, convex functions, subordination, fixed second coefficient.

condition:

$$\psi(i\rho,\sigma)\notin\Omega \quad \text{whenever} \quad (i\rho,\sigma)\in D \quad \text{and}$$

$$\sigma \leq -\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2), \quad \rho\in\mathbb{R},\ n\geq 1. \tag{1.2}$$

When $\Omega=\{w : \operatorname{Re} w>0\}$, $\Psi_{\mu,n}[\Omega]\equiv\Psi_{\mu,n}$. The following theorem is needed to prove our main results.

THEOREM 1.1. [**3**, Theorem 3.4] *Let $p\in\mathscr{H}_{\mu,n}$ with $0<\mu\leq 2$. Let $\psi\in\Psi_{n,\mu}$ with associated domain D. If $(p(z),zp'(z))\in D$ and $\operatorname{Re}\psi(p(z),zp'(z))>0$, then $\operatorname{Re}p(z)>0$ for $z\in\mathbb{D}$.*

For $0\leq\alpha,\ \beta\leq 1$, Ravichandran *et al.* [**10**] have shown that a function f of the form $f(z)=z+a_{n+1}z^{n+1}+\cdots$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right)>\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\beta-\frac{\alpha n}{2} \tag{1.3}$$

is starlike of order β. Several authors have investigated the functions satisfying (1.3), see [**4, 5, 12, 14**]. In the first result of Theorem 2.1, we obtain the corresponding result for $f\in\mathscr{A}_{n,b}$.

For function p of the form $p(z)=1+p_1z+p_2z^2+\cdots$, Nunokawa *et al.* [**8**] showed that for analytic function w, $\alpha p^2(z)+\beta zp'(z)\prec w(z)$ implies $\operatorname{Re}p(z)>0$, where $\beta>0$, $\alpha\geq-\beta/2$. See also [**11**]. Lemma 2.6 investigate the conditions for similar class of functions.

For complex numbers β and γ with $\beta\neq 0$, the differential subordination

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z),$$

where q is analytic and h is univalent with $q(0)=h(0)$, is popularly known as Briot-Bouquet differential subordination. This particular differential subordination has a significant number of important applications in the theory of analytic functions (for details see [**6**]). The importance of Briot-Bouquet differential subordination inspired many researchers to work in this area and many generalizations and extensions of the Briot-Bouquet differential subordination have recently been obtained. Ali *et al.* [**1**] obtained several results related to the Briot-Bouquet differential subordination. In Lemmas 2.2 and 2.5, the Briot-Bouquet differential subordination is investigated for functions with fixed second coefficient.

2. Sufficient conditions for starlikeness and univalence

Theorem 2.1 provides several sufficient conditions for starlikeness of order β, $0\leq\beta<1$ while Theorem 2.3 gives sufficient conditions for the inequality $\operatorname{Re}(f'(z))>\beta$ to hold. For $\beta=0$, this latter condition is sufficient for the close-to-convexity and hence univalence of the function f.

THEOREM 2.1. *Let $\alpha\geq 0$, $\beta\neq 1$, and $0\leq\mu\leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given by*

$$\delta_1=-\frac{\alpha}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+(1-\alpha)\beta+\alpha\beta^2,$$

$$\delta_2=-\frac{1}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta,$$

$$\delta_3 = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta, & if \quad 0<\beta\leq\frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta, & if \quad \frac{1}{2}\leq\beta, \end{cases}$$

$$\delta_4 = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad 0<\beta<\frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2}\leq\beta. \end{cases}$$

If $f\in\mathscr{A}_{n,b}$ satisfies one of the following subordinations

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\prec\frac{1+(1-2\delta_1)z}{1-z}, \tag{2.1}$$

$$\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec\frac{1+(1-2\delta_2)z}{1-z}, \tag{2.2}$$

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+(1-2\delta_3)z}{1-z}, \tag{2.3}$$

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\prec\frac{1+(1-2\delta_4)z}{1-z} \tag{2.4}$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{1+(1-2\beta)z}{1-z}.$$

THEOREM 2.2. *Let $\alpha\geq0$, $\beta\neq1$, and $0\leq\mu\leq2$. Let δ_1, δ_2, δ_3 and δ_4 be given as in Theorem 2.1. If $f\in\mathscr{A}_{n,b}$ satisfies one of the following subordinations*

$$-\frac{zf'(z)}{f(z)}\left(1-2\alpha-\alpha\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+(1-2\delta_1)z}{1-z}, \tag{2.5}$$

$$-\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+(1-2\delta_2)z}{1-z}, \tag{2.6}$$

$$-(1-\alpha)\frac{zf'(z)}{f(z)}-\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+(1-2\delta_3)z}{1-z}, \tag{2.7}$$

$$-1-\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}\prec\frac{1+(1-2\delta_4)z}{1-z} \tag{2.8}$$

then

$$-\frac{zf'(z)}{f(z)}\prec\frac{1+(1-2\beta)z}{1-z}.$$

THEOREM 2.3. *Let $\alpha\geq0$, $\beta\neq1$, and $0\leq\mu\leq2$. Let δ_1, δ_2, δ_3 and δ_4 be given as in Theorem 2.1. If $f\in\mathscr{A}_{n,b}$ satisfies one of the following subordinations*

$$f'(z)\left[\alpha\left(\frac{zf''(z)}{f'(z)}+f'(z)-1\right)+1\right]\prec\frac{1+(1-2\delta_1)z}{1-z}, \tag{2.9}$$

$$f'(z)+zf''(z)\prec\frac{1+(1-2\delta_2)z}{1-z}, \tag{2.10}$$

$$\alpha\frac{zf''(z)}{f'(z)}+f'(z)\prec\frac{1+(1-2\delta_3)z}{1-z}, \tag{2.11}$$

$$\frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\delta_4)z}{1-z} \tag{2.12}$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

The proof of these two theorems follows from the following series of lemmas.

LEMMA 2.1. *Let $\alpha \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 \leq \mu \leq 2$. For function $p \in \mathscr{H}_{\mu,n}$ and*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+(1-\alpha)\beta+\alpha\beta^2,$$

if p satisfies

$$(1-\alpha)p(z)+\alpha p^2(z)+\gamma zp'(z) \prec \frac{1+(1-2\delta)z}{1-z} \tag{2.13}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

PROOF. Define the function $q:\mathbb{D}\to\mathbb{C}$ by $q(z)=(p(z)-\beta)/(1-\beta)$. Then q is analytic and $(1-\beta)q(z)+\beta=p(z)$. By using this, the inequality (2.13) can then be written as

$$\operatorname{Re}\left[(1-\beta)(1-\alpha+2\alpha\beta)q(z)+\alpha(1-\beta)^2q^2(z)+\gamma(1-\beta)zq'(z)+(1-\alpha)\beta+\alpha\beta^2-\delta\right]>0.$$

Define the function $\psi:\mathbb{C}\to\mathbb{C}$ by

$$\psi(r,s)=(1-\beta)(1-\alpha+2\alpha\beta)r+\alpha(1-\beta)^2r^2+\gamma(1-\beta)s+(1-\alpha)\beta+\alpha\beta^2-\delta.$$

For $\rho\in\mathbb{R}$, $n\geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
&\operatorname{Re}\psi(i\rho,\sigma)\\
&=\operatorname{Re}\left[(1-\beta)(1-\alpha+2\alpha\beta)i\rho-\alpha(1-\beta)^2\rho^2+\gamma(1-\beta)\sigma+(1-\alpha)\beta+\alpha\beta^2-\delta\right]\\
&=\gamma(1-\beta)\sigma-\alpha(1-\beta)^2\rho^2+(1-\alpha)\beta+\alpha\beta^2-\delta\\
&\leq\gamma(1-\beta)\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]-\alpha(1-\beta)^2\rho^2+(1-\alpha)\beta+\alpha\beta^2-\delta\\
&=-\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)-\alpha(1-\beta)^2(\rho^2+1)+\alpha(1-\beta)^2\\
&\quad+(1-\alpha)\beta+\alpha\beta^2-\delta\\
&=-(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\alpha(1-\beta)^2\right]+\alpha(1-\beta)^2+(1-\alpha)\beta+\alpha\beta^2-\delta\\
&\leq-\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+(1-\alpha)\beta+\alpha\beta^2-\delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma)\leq 0$. By Theorem 1.1, $\operatorname{Re} q(z)>0$ or equivalently $\operatorname{Re} p(z)>\beta$. □

LEMMA 2.2. *Let $\alpha \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 \leq \mu \leq 2$. For function $p \in \mathscr{H}_{\mu,n}$ and*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\alpha\beta$$

if p satisfies

$$\alpha p(z)+\gamma z p'(z)\prec \frac{1+(1-2\delta)z}{1-z}$$

then

$$p(z)\prec \frac{1+(1-2\beta)z}{1-z}.$$

PROOF. First, replace $\alpha=0$ in Lemma 2.1, and then replace γ by γ/α to yield the result. $\square$

LEMMA 2.3. *Let $\alpha>0$, $\beta\neq 1$ and $0\leq\mu\leq 2$. Let*

$$\delta=\begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta:=\delta_2, & if \quad 0<\beta\leq\frac{1}{2},\\ \frac{-\alpha}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta:=\delta_1, & if \quad \frac{1}{2}\leq\beta, \end{cases}.$$

If the function $p\in\mathscr{H}_{\mu,n}$ satisfies

$$p(z)+\alpha\frac{zp'(z)}{p(z)}\prec\frac{1+(1-2\delta)z}{1-z} \tag{2.14}$$

then

$$p(z)\prec \frac{1+(1-2\beta)z}{1-z}.$$

PROOF. Similar to the proof of Lemma 2.1, let $q:\mathbb{D}\to\mathbb{C}$ be given by $q(z)=(p(z)-\beta)/(1-\beta)$. Then inequality (2.14) can be written as

$$\operatorname{Re}\left[(1-\beta)q(z)+\beta+\frac{\alpha(1-\beta)}{(1-\beta)q(z)+\beta}zq'(z)-\delta\right]>0. \tag{2.15}$$

Define the function $\psi:\mathbb{C}\to\mathbb{C}$ by

$$\psi(r,s)=(1-\beta)r+\frac{\alpha(1-\beta)}{(1-\beta)r+\beta}s+\beta-\delta.$$

Then $\operatorname{Re}\psi(q(z),zq'(z))>0$ and $\operatorname{Re}\psi(1,0)>0$. To show that $\psi\in\Psi_{\mu,n}$, by using (1.2), it follows that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma)&=\operatorname{Re}\left[(1-\beta)i\rho+\frac{\alpha(1-\beta)}{(1-\beta)i\rho+\beta}\sigma+\beta-\delta\right]\\ &=\operatorname{Re}\left[(1-\beta)i\rho+\frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma-\frac{\alpha(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma+\beta-\delta\right]\\ &=\frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma+\beta-\delta\\ &\leq\frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]+\beta-\delta\\ &=-\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right)+\beta-\delta.\end{aligned}$$

For $1/2\leq\beta$, the expression

$$\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}$$

attains minimum at $\rho = 0$ and therefore

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\beta^2}+\beta-\delta_1\\ &= \frac{-\alpha}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta_1.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma)\leq 0$.

For $0<\beta\leq 1/2$,

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(1-\beta)^2}+\beta-\delta_2\\ &= \frac{-\alpha\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta_2.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma)\leq 0$. Thus Theorem 1.1 implies $\operatorname{Re}q(z)>0$ or equivalently $\operatorname{Re}p(z)>\beta$. $\square$

LEMMA 2.4. *Let $\beta\neq 1$ and $0\leq\mu\leq 2$. Let*

$$\delta = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad 0<\beta<\frac{1}{2},\\ \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2}\leq\beta, \end{cases} \qquad (z\in\mathbb{D}).$$

If the function $p\in\mathscr{H}_{\mu,n}$ satisfies

$$\frac{zp'(z)}{p(z)}\prec\frac{1+(1-2\delta)z}{1-z}$$

then

$$p(z)\prec\frac{1+(1-2\beta)z}{1-z}.$$

PROOF. Let $q(z)=(p(z)-\beta)/(1-\beta)$ or $(1-\beta)q(z)+\beta=p(z)$. Then

$$\frac{zp'(z)}{p(z)}=\frac{(1-\beta)}{(1-\beta)q(z)+\beta}zq'(z). \tag{2.16}$$

Define $\psi:\mathbb{C}^2\to\mathbb{C}$ by

$$\psi(r,s)=\frac{(1-\beta)}{(1-\beta)r+\beta}s-\delta.$$

Then $\psi(r,s)$ is continuous on $\mathbb{C}-\{-\beta/(1-\beta)\}$ and by using (1.2), it follows that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left(\frac{(1-\beta)}{(1-\beta)i\rho+\beta}\sigma-\delta\right)\\ &= \operatorname{Re}\left(\frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma-\frac{(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma-\delta\right)\\ &= \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma-\delta\\ &\leq \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]-\delta\end{aligned}$$

$$= -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right)-\delta.$$

For $1/2 \leq \beta < 1$, the expression

$$\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}$$

attains its minimum at $\rho = 0$ and therefore

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\beta^2}-\delta\\ &= \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)-\delta.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$.

For $0 < \beta \leq 1/2$,

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(1-\beta)^2}-\delta\\ &= \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)-\delta.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$. Thus Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. $\square$

LEMMA 2.5. *Let* $\alpha > 0$, $\beta \neq 1$, *and* $0 \leq \mu \leq 2$. *Let*

$$\delta = \begin{cases} \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta, & if \quad \gamma \geq \alpha(1-2\beta),\\ \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta, & if \quad \gamma \leq \alpha(1-2\beta),\end{cases} \qquad (z \in \mathbb{D}).$$

If the function $p \in \mathscr{H}_{\mu,n}$ *satisfies*

$$p(z)+\frac{zp'(z)}{\alpha p(z)+\gamma} \prec \frac{1+(1-2\delta)z}{1-z}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

PROOF. Define $q(z) = (p-\beta)/(1-\beta)$ or $(1-\beta)q+\beta = p(z)$. Then

$$p(z)+\frac{zp'(z)}{\alpha p(z)+\gamma} = (1-\beta)q(z)+\beta+\frac{(1-\beta)}{\alpha[(1-\beta)q(z)+\beta]+\gamma}zq'(z). \tag{2.17}$$

Define $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = (1-\beta)r+\frac{(1-\beta)}{\alpha(1-\beta)r+\alpha\beta+\gamma}s+\beta-\delta.$$

Thus $\psi(r,s)$ is continuous and using (1.2), it follows that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left[(1-\beta)i\rho+\frac{(1-\beta)}{\alpha(1-\beta)i\rho+\alpha\beta+\gamma}\sigma+\beta-\delta\right]\\ &= \frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\sigma+\beta-\delta\end{aligned}$$

$$\leq \frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]+\beta-\delta$$
$$= \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\right)+\beta-\delta$$

For $\gamma \leq \alpha(1-2\beta)$,

$$\operatorname{Re}\psi(i\rho,\sigma) \leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\alpha^2(1-\beta)^2}+\beta-\delta$$
$$= \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta.$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$.

For $\gamma \geq \alpha(1-2\beta)$, the expression

$$\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}$$

attains minimum at $\rho = 0$ and therefore

$$\operatorname{Re}\psi(i\rho,\sigma) \leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(\alpha\beta+\gamma)^2}+\beta-\delta$$
$$= \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta.$$

Thus $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$ and result follows. □

LEMMA 2.6. *Let $\alpha \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 \leq \mu \leq 2$. If the function $p \in \mathscr{H}_{\mu,n}$ satisfies*

$$\alpha p^2(z)+\gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z} \quad (z \in \mathbb{D}) \tag{2.18}$$

where

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\alpha\beta^2$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

PROOF. Define $q(z) = (p(z)-\beta)/(1-\beta)$ or $(1-\beta)q(z)+\beta = p(z)$. Using this it can be shown that inequality (2.18) can be written as

$$\operatorname{Re}\left[\alpha((1-\beta)q(z)+\beta)^2+\gamma(1-\beta)zq'(z)-\delta\right] > 0.$$

Then $\psi(r,s)$ is define by

$$\psi(r,s) = \alpha[(1-\beta)r+\beta]^2+\gamma(1-\beta)s-\delta.$$

By using (1.2), it follows that

$$\begin{aligned}&\operatorname{Re}\psi(i\rho,\sigma)\\ &= \operatorname{Re}\left[\alpha((1-\beta)i\rho+\beta)^2+\gamma(1-\beta)\sigma-\delta\right]\\ &= -\alpha(1-\beta)\rho^2+\alpha\beta^2+\gamma(1-\beta)\sigma-\delta\end{aligned}$$

$$\begin{aligned}
&\leq \gamma(1-\beta)\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]+\alpha\beta^2-\alpha(1-\beta)\rho^2-\delta\\
&=-\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)-\alpha(1-\beta)^2(\rho^2+1)+\alpha(1-\beta)^2+\alpha\beta^2-\delta\\
&=-(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\alpha(1-\beta)^2\right]+\alpha(1-\beta)^2+\alpha\beta^2-\delta\\
&\leq -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\alpha\beta^2-\delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma)\leq 0$, and Theorem 1.1 implies $\operatorname{Re}q(z)>0$ or equivalently $\operatorname{Re}p(z)>\beta$. □

PROOF OF THEOREM 2.1. By taking $p(z)=zf'(z)/f(z)$, we have the following:

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)=p(z)+zp'(z),\\
&(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{zp'(z)}{p(z)}.
\end{aligned}$$

Hence the result follows from Lemmas 2.1, 2.2, 2.3 and 2.4. □

PROOF OF THEOREM 2.2. By taking $p(z)=-zf'(z)/f(z)$, we have the following:

$$\begin{aligned}
&-\frac{zf'(z)}{f(z)}\left(1-2\alpha-\alpha\frac{zf''(z)}{f'(z)}\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&-\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right)=p(z)+zp'(z),\\
&-(1-\alpha)\frac{zf'(z)}{f(z)}-\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&-1-\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}=\frac{zp'(z)}{p(z)}.
\end{aligned}$$

Hence the result follows from Lemmas 2.1, 2.2, 2.3 and 2.4. □

PROOF OF THEOREM 2.3. By taking $p(z)=f'(z)$, we have the following:

$$\begin{aligned}
&f'(z)\left(\alpha\left(\frac{zf''(z)}{f'(z)}+f'(z)-1\right)+1\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&f'(z)+\alpha zf''(z)=p(z)+\alpha zp'(z),\\
&\alpha\frac{zf''(z)}{f'(z)}+f'(z)=p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)}.
\end{aligned}$$

Hence the result follows from Lemmas 2.1, 2.2, 2.3 and 2.4. □

REMARK 2.1.

(i) *For* $\beta = 0$*, the condition* (2.9)–(2.12) *gives a sufficient condition for close-to-convexity and hence for univalence.*

(ii) *If* $\mu = 2$*, result* (2.1) *reduces to* [**10**, Theorem 2.1]*. If* $\mu = 2$*, and* $f'(z)$ *is considered as* $f(z)/z$*, result* (2.10) *reduces to* [**10**, Theorem 2.4]*. Inequality* (2.11) *reduces to* [**13**, Theorem 2, p. 182] *in the case when* $\mu = 2$*,* $n = 1$ *and* $\beta = 1/2$*. Furthermore, if* $\mu = 2$*,* $n = 1$ *and* $\beta = (\alpha + 1)/2$*, result* (2.12) *reduces to* [**9**, Theorem 1].

References

[1] R. M. Ali, V. Ravichandran and N. Seenivasagan, On Bernardi's integral operator and the Briot-Bouquet differential subordination, *J. Math. Anal. Appl.* **324** (2006), no. 1, 663–668.

[2] R. M. Ali, N. E. Cho, N. Jain and V. Ravichandran, Radii of starlikeness and convexity of functions defined by subordination with fixed second coefficients, *Filomat* **26** (2012), no. 3, 553-561.

[3] R. M. Ali, S. Nagpal and V. Ravichandran, Second-order differential subordination for analytic functions with fixed initial coefficient, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 3, 611–629.

[4] M.-S. Liu, Y.-Y. Liu and Z.-W. Liu, Properties and characteristics of certain subclass of analytic functions with positive coefficients, *Tamsui Oxford J. Inform. Math. Sci.* **27** (2011), no. 1, 39–60.

[5] M.-S. Liu, Y.-C. Zhu and H. M. Srivastava, Properties and characteristics of certain subclasses of starlike functions of order β, *Math. Comput. Modelling* **48** (2008), no. 3-4, 402–419.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Dekker, New York, 2000.

[7] S. Nagpal and V. Ravichandran, Applications of theory of differential subordination for functions with fixed initial coefficient to univalent functions, *Ann. Polon. Math.* **105** (2012), 225–238.

[8] M. Nunokawa, S. Owa, N. Takahashi and H. Saitoh, Sufficient conditions for Carathéodory functions, *Indian J. Pure Appl. Math.* **33** (2002), no. 9, 1385–1390.

[9] S. Owa, M. Nunokawa, H. Saitoh and H. M. Srivastava, Close-to-convexity, starlikeness, and convexity of certain analytic functions, Appl. Math. Lett. **15** (2002), no. 1, 63–69.

[10] V. Ravichandran, C. Selvaraj and R. Rajalaksmi, Sufficient conditions for starlike functions of order α, *JIPAM. J. Inequal. Pure Appl. Math.* **3** (2002), no. 5, Article 81, 6 pp.

[11] V. Ravichandran and M. Jayamala, On sufficient conditions for Caratheodory functions, *Far East J. Math. Sci. (FJMS)* **12** (2004), no. 2, 191–201.

[12] S. Singh and S. Gupta, A differential subordination and starlikeness of analytic functions, *Appl. Math. Lett.* **19** (2006), no. 7, 618–627.

[13] V. Singh, S. Singh and S. Gupta, A problem in the theory of univalent functions, *Integral Transforms Spec. Funct.* **16** (2005), no. 2, 179–186.

[14] D. Yang, S. Owa and K. Ochiai, Sufficient conditions for Carathéodory functions, *Comput. Math. Appl.* **51** (2006), no. 3-4, 467–474.

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